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ADMISSIBLE OBSERVATION OPERATORS FOR LINEAR SEMIGROUPS

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ABSTRACT
Consider a semigroup T on a Banach space X and a (possibly unbounded) operator C densely defined in X, with values in another Banach space. We give some necessary as well as some sufficient conditions for C to be an admissible observation operator for T, i.e., any finite segment of the output function $y(t) = C\mathbf{T}_t x$, $t \geqq 0$, should be in L^p and should depend continuously on the initial state x. Our approach is to start from a description of the map which takes initial states into output functions in terms of a functional equation. We also introduce an extension of C which permits a pointwise interpretation of $y(t) = C\mathbf{T}_t x$, even if the trajectory of x is not in the domain of C.

1. Introduction

Let $\mathbf{T} = (\mathbf{T}_t)_{t \geqq 0}$ be a strongly continuous semigroup on the Banach space X, let Y be another Banach space, let $C: W \to Y$ be a linear operator, where W is a dense T-invariant subspace of X and let $p \in [1, \infty]$. In this paper we investigate the concept of admissibility of C as an observation operator for T and p, which means that the formula

$$(1.1) \qquad y(t) = C\mathbf{T}_t x, \qquad \text{for } t \geqq 0,$$

defines a continuous map from X to Y-valued functions of class L^p_{loc}.

Before giving the precise interpretation of (1.1), let us say some words about the intuitive meaning and the importance of our problem. In linear systems theory, one usually deals with systems described by the equations

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$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t). \end{cases}$$

Here $x(t)$ is the state of the system at time t, $u(t)$ is the input and $y(t)$ is the output. We have $x(t) \in X$, the Banach space X being the state space, $u(t) \in U$, the Banach space U being the input space, $y(t) \in Y$, the Banach space Y being the output space, and A is the generator of a strongly continuous semigroup $\mathbf{T}$ on X. The input function $u(\cdot)$ and the output function $y(\cdot)$ are assumed to be locally L^p, for some $p \in [1, \infty]$. The control operator B and the observation operator C may be unbounded, in which case the well-posedness of such a system (i.e. continuous dependence, for any $t > 0$, of $x(t) \in X$ and $y(\cdot) \in L^p([0, t], Y)$ on $x(0) \in X$ and $u(\cdot) \in L^p([0, t], U)$) is not easy to establish. In the paper Weiss [18] we have analysed the particular case $C = 0$ (i.e. we have ignored the output). In this paper we restrict ourselves to the particular case $B = 0$ (i.e. we consider there is no input). Some of the results for $B = 0$ can be obtained from those for $C = 0$ by duality, but most can not (for details on that, see Section 6). The results which we get are needed for the treatment of the general case (the well-posedness of the triple (A, B, C)), as will be hinted in Sections 4 and 7 and worked out in detail in the article [21].

Unbounded observation operators appear naturally, for example when one models boundary or point observation for systems described by linear P.D.E.'s. There is an extensive literature dealing with systems having unbounded observation operators, for example Curtain and Pritchard [2, chapter 8], Fuhrmann [6, chapter III], Lasiecka and Triggiani [8], Pritchard and Salamon [11], Pritchard and Townley [12], Pritchard and Wirth [13], Salamon [14], [15], [16], Seidman [17], Yamamoto [20].

The *standard interpretation* given to (1.1) is the following. This formula has a clear meaning for $x \in W$. If the functions $y(\cdot)$ obtained in this way are locally in L^p then for any fixed $T > 0$ we define the operator $\mathbf{L}_T : W \to L^p([0, \infty), Y)$ by truncating $y(\cdot)$ to $[0, T)$

$$(\mathbf{L}_T x)(t) = \begin{cases} C\mathbf{T}_t x, & \text{for } t \in [0, T), \\ 0, & \text{for } t \geqq T. \end{cases} \tag{1.2}$$

(We don't consider $\mathbf{L}_T : W \to L^p([0, T], Y)$ to avoid complications later, which would occur if the range space of $\mathbf{L}_T$ would depend on T.) If $\mathbf{L}_T$ has a continuous extension to all of X (denoted by the same symbol) then we say that (for the semigroup $\mathbf{T}$ and the index p) the observation operator C is *admissible* (it is easy to show that this doesn't depend on the choice of T). In this case, we interpret (1.1) as meaning

$$y(t) = (\mathbf{L}_T x)(t) \qquad \text{for } t \in [0, T),$$

which makes sense even if $\mathbf{T}_t x$ is never in W. (The two sides of the above equality are of course defined only up to a null-set.) By choosing larger and larger numbers T, we get the function $y(\cdot)$ for all $t \geqq 0$ (up to a null-set). This function we call *the output function corresponding to* x (see also the end of Section 2).

Admissible observation operators which yield the same operators $\mathbf{L}_T$ (i.e. the same outputs) will be called *equivalent*. It may happen that operators having domains whose intersection is zero are equivalent; see Example 1.2 below. But even if two equivalent observation operators have the same domain, they do not have to coincide on it; see again Example 1.2.

We show that the above interpretation of (1.1) can be replaced by an (equivalent) *pointwise interpretation*, like for bounded C. We prove the following. Let $A : D(A) \to X$ be the generator of $\mathbf{T}$. Let X_1 denote the space $D(A)$ with the graph norm. If C is admissible then it is equivalent to an operator in $\mathscr{L}(X_1, Y)$. Therefore we may regard admissible observation operators as elements of $\mathscr{L}(X_1, Y)$. Therefore we may regard admissible observation operators as elements of $\mathscr{L}(X_1, Y)$ (they form a subspace of $\mathscr{L}(X_1, Y)$). Every admissible $C \in \mathscr{L}(X_1, Y)$ has an extension to an operator C_L defined on a Banach space $D(C_L)$ (continuously embedded in X) which has the following property: for any $x \in X$ we have that for almost every $t \geqq 0$, $\mathbf{T}_t x \in D(C_L)$ and $y(t) = C_L \mathbf{T}_t x$, where $y(\cdot)$ is the output function corresponding to x. Thus, replacing in (1.1) C by C_L, the formula makes sense for any $x \in X$ and a.e. $t \geqq 0$, which is the best we can hope for, since functions in L^p_{loc} are defined only up to a null-set.

We outline the contents of the following sections. In §2 we introduce the concept of an *abstract linear observation system*, which is motivated as follows. Suppose C is admissible for $\mathbf{T}$. Then $\mathbf{L} = (\mathbf{L}_t)_{t \geqq 0}$ (as defined by (1.2) and continuous extension) is a family of bounded linear operators from X to $L^p([0, \infty), Y)$. The semigroup $\mathbf{T}$ and the family $\mathbf{L}$ satisfy a natural functional equation, see equation (2.1) below, which we call the *composition property*. We define an abstract linear observation system as a pair $(\mathbf{L}, \mathbf{T})$, where $\mathbf{T}$ is a strongly continuous semigroup and $\mathbf{L}$ is a family of operators such that the composition property holds. This is a simple and natural concept, in whose definition no mention of unbounded operators or of dense T-invariant subspaces is needed.

In §3 we prove a representation theorem which states that any abstract linear observation system is described by (1.1) (with the standard interpretation), with $C \in \mathscr{L}(X_1, Y)$. That implies that any admissible observation operator is equivalent to an element of $\mathscr{L}(X_1, Y)$.

In §4 we define the operator C_L, which we call the *Lebesgue extension of* C, and show that C_L enables us to replace the standard interpretation of (1.1) by the pointwise one. We also indicate other applications of C_L.

In §5 we show that C_L remains unchanged if the generator of the semigroup is perturbed by a bounded operator.

In §6 we introduce the Banach space of admissible observation operators C for given X, Y, T and p, denoted $\mathscr{C}_p$. We also discuss the analogy between the theory of admissible control operators and that of admissible observation operators.

In §7 we deal with the particular case of diagonal semigroups on l^2 with scalar output. We mention a Carleson measure criterion for admissibility and give a method for computing $C_L x$.

Now we give two simple examples to illustrate the theory. We will return to them in the following sections.

EXAMPLE 1.1. Consider the observed heat equation on $[0, \pi]$

$$\begin{cases} \dfrac{\partial}{\partial t}\psi(\zeta, t) = \dfrac{\partial^2}{\partial \zeta^2}\psi(\zeta, t), \\ \psi(0, t) = \psi(\pi, t) = 0, \\ \psi(\zeta, 0) = \psi_0(\zeta), \\ y(t) = \psi(\xi, t), \end{cases}$$

where $\xi \in (0, \pi)$ is fixed and $\psi_0 \in L^2[0, \pi]$.

Let $X = L^2[0, \pi]$. Let us denote $A = d^2/d\zeta^2$, with domain

$$D(A) = \{x \in X \mid x, x' \in \mathrm{AC}[0, \pi], x'' \in L^2[0, \pi], x(0) = x(\pi) = 0\}$$

(the letters AC stand for absolutely continuous). Then A generates an analytic and compact semigroup T on X. Taking $C = \delta_\xi$ (the point-evaluation operator in the point ξ), with domain $W = C[0, \pi]$, our system is described by (1.1). C is admissible for $p < 4$; see Curtain and Pritchard [2, p. 216].

EXAMPLE 1.2. Let $r \in [1, \infty)$. Let X be the closed subspace of $L^r[0, 2\pi]$ defined by

$$X = \left\{ x \in L^r[0, 2\pi] \,\middle|\, \int_0^{2\pi} x(\zeta)d\zeta = 0 \right\}.$$

Let $\mathbf{T}$ be the semigroup of periodic left shifts on X, i.e.

$$(\mathbf{T}_t z)(\zeta) = z(\zeta + t - k \cdot 2\pi), \qquad \text{for } k \cdot 2\pi \leqq \zeta + t < (k+1) \cdot 2\pi.$$

By a step function on $[0, 2\pi]$ we mean a function constant on each of a finite set of nonoverlapping intervals covering $[0, 2\pi]$. Let W_1 be the vector space of step functions contained in X, let $W_2 = W_1$ and let W_3 be the vector space of trigonometric polynomials contained in X. Let for $i \in \{1, 2, 3\}$, $C_i : W_i \to \mathbf{C}$ be defined by

$$C_1 x = x(0)$$

(i.e. the value of x on the first interval of constancy),

$$C_2 x = x(2\pi)$$

(i.e. the value of x on the last interval of constancy) and

$$C_3 x = x(0).$$

Then for $p \leqq r$, C_1, C_2 and C_3 are admissible and equivalent, despite the facts that C_1 and C_2 do not coincide on their (common) domain and $W_1 \cap W_3 = \{0\}$.

2. Abstract linear observation systems

We begin by giving the formal definition of an abstract linear observation system, as announced in §1. For that we need the notion of *concatenation* on $\Gamma = L^p([0, \infty), Y)$, where Y is a Banach space.

Let $y, v \in \Gamma$ and let $\tau \geqq 0$. Then the *τ-concatenation of y and v*, $y \underset{\tau}{\Diamond} v \in \Gamma$, is given by

$$\left(y \underset{\tau}{\Diamond} v\right)(t) = \begin{cases} y(t) & \text{for } t \in [0, \tau), \\ v(t - \tau) & \text{for } t \geqq \tau. \end{cases}$$

Recall that we work with Γ because we want to define our system as transmitting Y-valued locally p-integrable output functions and any segment of such an output function can be thought of as the restriction to a bounded interval of an element of Γ.

DEFINITION 2.1. Let X and Y be Banach spaces, $p \in [1, \infty]$ and $\Gamma = L^p([0, \infty), Y)$.

An *abstract linear observation system* on X and Γ is a pair $(\mathbf{L}, \mathbf{T})$, where $\mathbf{T} = (\mathbf{T}_t)_{t \geqq 0}$ is a strongly continuous semigroup on X and $\mathbf{L} = (\mathbf{L}_t)_{t \geqq 0}$ is a family of bounded operators from X to Γ such that

$$\mathbf{L}_{\tau+t}x = \mathbf{L}_\tau x \underset{\tau}{\diamondsuit} \mathbf{L}_t \mathbf{T}_\tau x \tag{2.1}$$

for any $x \in X$ and any $\tau, t \geqq 0$, and $\mathbf{L}_0 = 0$.

The functional equation (2.1) is called the *composition property*. The operators $\mathbf{L}_t$ are called *output maps*.

The above definition is motivated as follows. Let X, $\mathbf{T}$, W, Y, C and p be as in the introduction and let C be admissible, as defined there. Then C and $\mathbf{T}$ define, via (1.2) and continuous extension, a family of operators $\mathbf{L}$ such that $(\mathbf{L}, \mathbf{T})$ is an abstract linear observation system.

REMARK 2.2. Taking in (2.1) $t = 0$, we get that for any $\tau \geqq 0$

$$\mathbf{L}_\tau = \mathbf{P}_\tau \mathbf{L}_\tau, \tag{2.2}$$

where $\mathbf{P}_\tau$ is the projection of Γ defined by truncation to $[0, \tau)$

$$\mathbf{P}_\tau y = y \underset{\tau}{\diamondsuit} 0.$$

From (2.1) and (2.2) we immediately get that for $T \geqq \tau$

$$\mathbf{L}_\tau = \mathbf{P}_\tau \mathbf{L}_T. \tag{2.3}$$

In particular, $\| \mathbf{L}_t \|$ is nondecreasing with t.

We give an estimate for the growth rate of $\| \mathbf{L}_t \|$.

PROPOSITION 2.3. *Let X and Γ be as in Definition* 2.1 *and let* $(\mathbf{L}, \mathbf{T})$ *be an abstract linear observation system on X and* Γ.

If $M \geqq 1$ and $\omega > 0$ are such that

$$\| \mathbf{T}_t \| \leqq M e^{\omega t}, \qquad \forall t \geqq 0, \tag{2.4}$$

then there is some $L \geqq 0$ such that

$$\| \mathbf{L}_t \| \leqq L e^{\omega t}, \qquad \forall t \geqq 0. \tag{2.5}$$

PROOF. It is easy to show by induction that for any $n \in \mathbf{N}$

$$\| \mathbf{L}_n x \| = (\| \mathbf{L}_1 x \|^p + \| \mathbf{L}_1 \mathbf{T}_1 x \|^p + \cdots + \| \mathbf{L}_1 \mathbf{T}_{n-1} x \|^p)^{1/p}$$
$$\leqq \| \mathbf{L}_1 x \| + \| \mathbf{L}_1 \mathbf{T}_1 x \| + \cdots + \| \mathbf{L}_1 \mathbf{T}_{n-1} x \|$$

(for $p = \infty$ the argument has to be written in a slightly different form), whence by (2.4)

$$\| \mathbf{L}_n \| \leqq [1 + Me^{\omega} + \cdots + Me^{\omega(n-1)}] \cdot \| \mathbf{L}_1 \|$$
$$\leqq M \frac{e^{\omega n} - 1}{e^{\omega} - 1} \| \mathbf{L}_1 \| .$$

Denoting

$$L = M \frac{e^{\omega}}{e^{\omega} - 1} \| \mathbf{L}_1 \|$$

we get for $t \in [n-1, n)$

$$\| \mathbf{L}_t \| \leqq \| \mathbf{L}_n \| \leqq L e^{\omega(n-1)} \leqq L e^{\omega t},$$

i.e. estimate (2.5). □

REMARK 2.4. For $\omega = 0$, $\mathbf{L}$ does not have to be uniformly bounded but one can easily obtain that

$$\| \mathbf{L}_t \| \leqq L \cdot (1 + t)^{1/p}, \qquad \forall t \geqq 0,$$

for some $L \geqq 0$. For $\omega < 0$, $\mathbf{L}$ is uniformly bounded.

Let $\tilde{\Gamma} = L^p_{loc}([0, \infty), Y)$. Concatenation and the projections $\mathbf{P}_\tau$ have obvious extensions to $\tilde{\Gamma}$. $\tilde{\Gamma}$ is a Fréchet space with the family of seminorms $p_n(y) = \| P_n y \|$, $n \in \mathbf{N}$. We have $\Gamma \subset \tilde{\Gamma}$ densely, with continuous embedding. It is easy to check that for any $x \in X$, $\mathbf{L}_\tau x$ is convergent in $\tilde{\Gamma}$, as $\tau \to \infty$ (if $\mathbf{T}$ is exponentially stable then this is true even in Γ). Let

$$\mathbf{L}_\infty x = \lim_{\tau \to \infty} \mathbf{L}_\tau x.$$

Then $\mathbf{L}_\infty \in \mathscr{L}(X, \tilde{\Gamma})$ and (2.3) extends to

$$\mathbf{L}_\tau = \mathbf{P}_\tau \mathbf{L}_\infty, \tag{2.6}$$

valid for any $\tau \geqq 0$. For any $x \in X$, $\mathbf{L}_\infty x$ is called *the output function corresponding to* x. The functional equation (2.1) together with (2.6) imply that

$$\mathbf{L}_\infty x = \mathbf{L}_\infty x \underset{\tau}{\diamondsuit} \mathbf{L}_\infty \mathbf{T}_\tau x, \tag{2.7}$$

for any $x \in X$ and any $\tau \geqq 0$.

3. The representation theorem

DEFINITION 3.1. Let X be a Banach space and $\mathbf{T}$ a strongly continuous semigroup on X with generator $A : D(A) \to X$. Let $\beta \in \rho(A)$, the resolvent set of A (if X is real, take $\beta \in \mathbf{R}$). We define the space X_1 to be $D(A)$ with the norm

$$\| x \|_1 = \| (\beta I - A)x \| ,$$

and the space X_{-1} to be the completion of X with respect to the norm

$$\| x \|_{-1} = \| (\beta I - A)^{-1}x \| .$$

REMARK 3.2. It is easy to verify that for any different $\beta_1 \in \rho(A)$ instead of β we get equivalent norms $\| \cdot \|_1$ and $\| \cdot \|_{-1}$ (so X_{-1} does not depend on β). Further $\| \cdot \|_1$ is equivalent with the graph norm on $D(A)$, so X_1 is complete.

The spaces X_1, X_{-1} appear for example in Nagel et al. [9, p. 19], Salamon [14] and Weiss [18], where their properties are discussed.

The next theorem is a reformulation and slight generalization of a part of a theorem of D. Salamon (see Remark 3.6 below). We give here a (new) proof for the sake of completeness.

THEOREM 3.3. *Let X and Y be Banach spaces, $p \in [1, \infty]$ and $\Gamma = L^p([0, \infty), Y)$. Let* $(\mathbf{L}, \mathbf{T})$ *be an abstract linear observation system on X and Γ.*

Then there is a unique $C \in \mathcal{L}(X_1, Y)$ such that for any $x \in X_1$ and any $t \geqq 0$

$$(\mathbf{L}_\infty x)(t) = C\mathbf{T}_t x. \tag{3.1}$$

The above result can be called a representation theorem, because formula (3.1) completely determines $\mathbf{L}_\infty$, since X_1 is dense in X. The family $\mathbf{L}$ is determined by $\mathbf{L}_\infty$ through (2.6). C is called the *observation operator of the system* $(\mathbf{L}, \mathbf{T})$.

PROOF. Let $L \geqq 0$ and $\omega > 0$ be such that (2.5) holds. Let for any $s \in \mathbf{C}$ with $\operatorname{Re} s > \omega$ the operator $\Lambda_s : X \to Y$ be defined by the Laplace-integral

$$\Lambda_s x = \int_0^\infty e^{-st}(\mathbf{L}_\infty x)(t)dt.$$

We have to check that this definition is correct, i.e. the above integral converges absolutely. We have, using (2.5) and (2.6) and denoting $\lambda = \operatorname{Re} s$,

$$\int_0^\infty \| e^{-st}(\mathbf{L}_\infty x)(t) \| \, dt = \sum_{n=1}^{\infty} \int_{n-1}^{n} e^{-\lambda t} \| (\mathbf{L}_\infty x)(t) \| \, dt$$

$$\leqq e^{\lambda} \sum_{n=1}^{\infty} e^{-\lambda n} \| \mathbf{L}_n x \|$$

$$\leqq L e^{\lambda} \sum_{n=1}^{\infty} e^{-(\lambda - \omega)n} \| x \| .$$

(We have used above that on $[n-1, n]$, the L^1-norm is smaller or equal the L^p-norm.) Thus we have got that for $\operatorname{Re} s > \omega$, Λ_s is well defined and moreover $\Lambda_s \in \mathcal{L}(X, Y)$.

The functional equation (2.7) implies that for any $x \in X$ and any $\tau > 0$

$$\Lambda_s x = \int_0^\tau e^{-st}(\mathbf{L}_\infty x)(t) dt + \int_\tau^\infty e^{-st}(\mathbf{L}_\infty \mathbf{T}_\tau x)(t-\tau) dt$$

$$= \int_0^\tau e^{-st}(\mathbf{L}_\infty x)(t) dt + e^{-s\tau} \Lambda_s \mathbf{T}_\tau x .$$

Rearranging we have

$$\text{(3.2)} \qquad \frac{1}{\tau} \int_0^\tau e^{-st}(\mathbf{L}_\infty x)(t) dt = \frac{1 - e^{-s\tau}}{\tau} \Lambda_s x - e^{-s\tau} \Lambda_s \frac{\mathbf{T}_\tau x - x}{\tau} .$$

For $x \in D(A)$ the right-hand side of (3.2) converges as $\tau \to 0$, so the left-hand side has to converge too. Moreover, the limit doesn't depend on s, because of the simple fact that

$$\text{(3.3)} \qquad \lim_{\tau \to 0} \left[\frac{1}{\tau} \int_0^\tau e^{-st}(\mathbf{L}_\infty x)(t) dt - \frac{1}{\tau} \int_0^\tau (\mathbf{L}_\infty x)(t) dt \right] = 0.$$

Let us denote for $x \in D(A)$

$$\text{(3.4)} \qquad Cx = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau (\mathbf{L}_\infty x)(t) dt .$$

Then (3.2) and (3.3) imply that for $x \in D(A)$

$$\text{(3.5)} \qquad Cx = s \Lambda_s x - \Lambda_s A x ,$$

and since $A \in \mathcal{L}(X_1, X)$, we get that

$$C \in \mathcal{L}(X_1, Y).$$

Denoting $z = (sI - A)x$, (3.5) can be written in the form

$$(3.6) \qquad \Lambda_s z = C(sI - A)^{-1}z,$$

which holds for any $z \in X$, because $sI - A$ maps $D(A)$ onto X.

Let $\mathcal{Y}$ be the space of those strongly measurable functions $y:[0,\infty) \to Y$ whose Laplace-integral is absolutely convergent for $\operatorname{Re} s > \omega$ (we identify functions which are equal a.e.). We have seen at the beginning of the proof that $\mathbf{L}_\infty z \in \mathcal{Y}$ for any $z \in X$. On the other hand, for $x \in D(A)$, the function $\eta_x : [0,\infty) \to Y$ defined by $\eta_x(t) = C\mathbf{T}_t x$ belongs to $\mathcal{Y}$. This follows from the fact that $\mathbf{T}$ is a strongly continuous semigroup on X_1, satisfying the same inequality (2.4) as on X, and $C \in \mathcal{L}(X_1, Y)$. Since the Laplace transformation is one-to-one on $\mathcal{Y}$, it follows from (3.6) that $\mathbf{L}_\infty x = \eta_x$, i.e. (3.1). The uniqueness of C is obvious. □

REMARK 3.4. It follows from (3.1) that for $x \in D(A)$, $\mathbf{L}_\infty x$ is continuous on $[0,\infty)$ and the operator C has the following expression in terms of $\mathbf{L}_\infty$:

$$(3.7) \qquad Cx = (\mathbf{L}_\infty x)(0).$$

In fact, for $x \in D(A)$ the function $\mathbf{L}_\infty x$ is more than just continuous. To elaborate on that, we recall that a function $y:[0,\infty) \to Y$ is said to be of class $W^{1,p}_{\mathrm{loc}}$ if there is a function $\gamma \in L^p_{\mathrm{loc}}([0,\infty), Y)$ such that for any $t \geqq 0$

$$y(t) = y(0) + \int_0^t \gamma(\sigma)d\sigma.$$

In particular, y is absolutely continuous.

PROPOSITION 3.5. *In the conditions of Theorem* 3.3, *for any* $x \in D(A)$, *the function* $\mathbf{L}_\infty x$ *is of class* $W^{1,p}_{\mathrm{loc}}$.

PROOF. Let $x \in D(A^2)$. By (3.1) we have

$$(\mathbf{L}_\infty Ax)(t) = C\mathbf{T}_t Ax,$$

and after integrating both sides

$$(3.8) \qquad C\mathbf{T}_t x = Cx + \int_0^t (\mathbf{L}_\infty Ax)(\sigma)d\sigma.$$

Both sides of (3.8) depend continuously on x as an element of X_1, and since $D(A^2)$ is dense in X_1, (3.8) remains valid for $x \in D(A)$. By (3.1) and the fact that $\mathbf{L}_\infty Ax \in L^p_{\mathrm{loc}}([0,\infty), Y)$ we get that $\mathbf{L}_\infty x$ is of class $W^{1,p}_{\mathrm{loc}}$. □

REMARK 3.6. Theorem 3.3 is contained in a representation theorem of Salamon [15], which concerns systems having input, state and output.

Actually, he considers only the case when X and Y are Hilbert spaces and $p = 2$, but the part of his proof which concerns the operator C can be easily rewritten for our more general X, Y and p. His proof doesn't make use of Laplace transforms; he first proves what appears above as Proposition 3.5 and then defines C through (3.7).

REMARK 3.7. The observation operator C which appears in Theorem 3.3 is obviously admissible for $\mathbf{T}$ and p, as defined in the introduction. With the commentary following Definition 2.1 we conclude that any admissible observation operator for $\mathbf{T}$ and p is equivalent (as defined in the introduction) to an operator in $\mathcal{L}(X_1, Y)$.

4. The Lebesgue extension

DEFINITION 4.1. Let X, Y be Banach spaces, $\mathbf{T}$ a semigroup on X and $C \in \mathcal{L}(X_1, Y)$. We define the operator $C_L : D(C_L) \to Y$, the *Lebesgue extension of* C (with respect to $\mathbf{T}$), by

$$C_L x = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \mathbf{T}_\sigma x \quad d\sigma, \tag{4.1}$$

where

$$D(C_L) = \{x \in X \mid \text{the limit in (4.1) exists}\}.$$

We define on $D(C_L)$ the norm

$$\| x \|_{D(C_L)} = \| x \| + \sup_{\tau \in (0,1]} \left\| C \frac{1}{\tau} \int_0^\tau \mathbf{T}_\sigma x \, d\sigma \right\| .$$

The name "Lebesgue extension" will be justified in Theorem 4.5 below.

REMARK 4.2. If A, the generator of $\mathbf{T}$, is invertible then

$$C_L x = \lim_{\tau \to 0} C \frac{\mathbf{T}_\tau - I}{\tau} A^{-1} x. \tag{4.2}$$

PROPOSITION 4.3. *With the notation of Definition* 4.1, $D(C_L)$ *is a Banach space, we have*

$$X_1 \subset D(C_L) \subset X$$

with continuous embeddings, and $C_L \in \mathcal{L}(D(C_L), Y)$.

PROOF. It follows easily from the fact that $\mathbf{T}$ is a strongly continuous semigroup on X_1, that $X_1 \subset D(C_L)$, with continuous embedding. The fact that $D(C_L) \subset X$, with continuous embedding, is trivial.

Let the operator $M : D(C_L) \to C([0, 1], Y)$ be defined by

$$(Mx)(\tau) = \begin{cases} C \dfrac{1}{\tau} \displaystyle\int_0^\tau \mathbf{T}_\sigma x \, d\sigma, & \text{for } \tau \in (0, 1], \\ C_L x, & \text{for } \tau = 0. \end{cases}$$

Let us check that the definition of M is correct. For any $x \in X$ the function

$$m(\tau) = \int_0^\tau \mathbf{T}_\sigma x \, d\sigma$$

belongs to $C([0, 1], X_1)$, so $C(1/\tau)m(\tau)$, as a function of τ, is continuous on $(0, 1]$. For $x \in D(C_L)$ we also have continuity in 0, by the definition of C_L. Therefore M is well defined.

It is not difficult to check that, as an operator densely defined in X, M is closed. Since the norm on $D(C_L)$ is in fact the graph norm with respect to M, it follows that $D(C_L)$ is a Banach space and that M is a bounded linear operator from $D(C_L)$ to $C([0, 1], Y)$. From $C_L x = (Mx)(0)$ we get that $C_L \in \mathscr{L}(D(C_L), Y)$. □

Remark 4.4. The space $D(C_L)$ with the graph norm of C_L, i.e. $|||x||| = \|x\| + \|C_L x\|$, is usually not complete, because C_L is usually not closed as an operator densely defined in X. In fact, if C_L is not bounded on X and if Y is finite dimensional, then C_L can not be closable.

Let $y \in L^1_{\mathrm{loc}}([0, \infty), Y)$ and let $t \geqq 0$. We say that y has a *Lebesgue point* in t if the limit

$$\tilde{y}(t) = \lim_{\tau \to 0} \frac{1}{\tau} \int_t^{t+\tau} y(\sigma) d\sigma$$

exists. It is well known that almost every $t \geqq 0$ is a Lebesgue point of y and we have $\tilde{y}(t) = y(t)$ for a.e. $t \geqq 0$ (see Diestel and Uhl [3, p. 49]).

Theorem 4.5. *Let X and Y be Banach spaces, $p \in [1, \infty]$ and $\Gamma = L^p([0, \infty), Y)$. Let $(\mathbf{L}, \mathbf{T})$ be an abstract linear observation system on X and Γ. Let C be the observation operator of the above system and let C_L be its Lebesgue extension.*

Let $x \in X$. For any $t \geqq 0$, we have that $\mathbf{T}_t x \in D(C_L)$ if and only if $\mathbf{L}_\infty x$ has a Lebesgue point in t. In particular,

$$\mathbf{T}_t x \in D(C_L) \qquad \textit{for a.e. } t \geqq 0.$$

Further, we have

$$(4.3) \qquad (\mathbf{L}_\infty x)(t) = C_L \mathbf{T}_t x \qquad \textit{for a.e. } t \geqq 0.$$

Taking in the first part of the above theorem $t = 0$, we get that $x \in D(C_L)$ if and only if $\mathbf{L}_\infty x$ has a Lebesgue point in 0.

PROOF. We have for any $x \in X$, any $t \geqq 0$ and any $\tau > 0$

$$(4.4) \qquad \frac{1}{\tau} \int_t^{t+\tau} (\mathbf{L}_\infty x)(\sigma) d\sigma = C \frac{1}{\tau} \int_0^\tau \mathbf{T}_\sigma(\mathbf{T}_t x) d\sigma.$$

Indeed, (4.4) holds for $x \in X_1$ by (3.1) and the fact that $C \in \mathscr{L}(X_1, Y)$. Both sides of (4.4) depend continuously on x as an element of X, so by the density of X_1 in X we get (4.4).

Taking in (4.4) $\tau \to 0$, we get that $\mathbf{T}_t x \in D(C_L)$ if and only if $\mathbf{L}_\infty x$ has a Lebesgue point in t. By the result on Lebesgue points mentioned before the statement of the theorem, we get that for almost every $t \geqq 0$, $\mathbf{T}_t x \in D(C_L)$ and (4.3) holds. □

REMARK 4.6. Taking in (4.4) $t = 0$ and comparing with (4.1), we get that for C as in Theorem 4.5 and $x \in D(C_L)$

$$C_L x = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau (\mathbf{L}_\infty x)(\sigma) d\sigma.$$

If x is not in $D(C_L)$, then the limit above is not convergent.

PROPOSITION 4.7. *With the notation of Theorem* 4.5, *we have for* $x \in D(C_L)$

$$C_L x = \lim_{\lambda \to \infty} C\lambda(\lambda I - A)^{-1} x,$$

where A is the generator of $\mathbf{T}$.

PROOF. By Remark 4.6, for any $\tau \geqq 0$

$$\int_0^\tau (\mathbf{L}_\infty x)(\sigma) d\sigma = \tau(C_L x + \delta(\tau)),$$

where $\delta(\tau) \to 0$ as $\tau \to 0$. Applying the Laplace transformation and using (3.6) we get for $\lambda \in \mathbf{R}$ sufficiently big

$$\frac{1}{\lambda}\,C(\lambda I - A)^{-1}x = \frac{1}{\lambda^2}\,C_L x + \int_0^\infty e^{-\lambda\tau}\tau\delta(\tau)d\tau,$$

whence

$$\| C_L x - C\lambda(\lambda I - A)^{-1}x \| \leqq \lambda^2 \int_0^\infty e^{-\lambda\tau}\tau \,\| \delta(\tau) \|\, d\tau.$$

It is easy to show that the right-hand side above tends to 0 as $\lambda \to \infty$. □

I do not know if it is possible that the limit in the above proposition converges for an $x \in X$ which is not in $D(C_L)$.

REMARK 4.8. In Proposition 4.7, instead of real λ one may take complex λ in an angular sector defined by $|\arg \lambda| \leqq \psi$, where $\psi < \pi/2$.

REMARK 4.9. Let us say a few words about the importance of the Lebesgue extension for linear systems having input, state and output. A detailed treatment of this subject will appear in the article [22].

Let X, Y, $\mathbf{T}$, p, C and C_L be as in Theorem 4.5. Let U be a Banach space and let $B \in \mathscr{L}(U, X_{-1})$ be an admissible control operator for $\mathbf{T}$ and p (see Weiss [18]). Assume that for some (and hence all) $s \in \rho(A)$, the range of $(sI - A)^{-1}B$ is contained in $D(C_L)$. Then

$$\mathbf{H}(s) = C_L(sI - A)^{-1}B$$

is an analytic $\mathscr{L}(U, Y)$-valued function on $\rho(A)$. $\mathbf{H}(s)$ is the *transfer function* of the triple (A, B, C) in the following sense: For any U-valued step function of compact support u, the mild solution of $\dot{x}(t) = Ax(t) + Bu(t)$ is a.e. in $D(C_L)$, $y(t) = C_L x(t)$ is in $L^p_{loc}([0, \infty), Y)$, it has a Laplace transform $\hat{y}$ and $\hat{y}(s) = \mathbf{H}(s)\hat{u}(s)$. The triple (A, B, C) is well posed if the above map from step functions to L^p_{loc} has a continuous extension to a map from L^p_{loc} to L^p_{loc}. In that case, even for $u \in L^p_{loc}([0, \infty), U)$, the mild solution of $\dot{x}(t) = Ax(t) + Bu(t)$ is a.e. in $D(C_L)$ and $C_L x(\cdot)$ is in $L^p_{loc}([0, \infty), Y)$. (See also Remark 7.3 below.)

REMARK 4.10. As already mentioned in the introduction, Theorem 4.5 enables us to give a pointwise interpretation to (1.1), after replacing C by C_L. We mention that the equation $\dot{x}(t) = Ax(t)$, which describes the evolution of the state, also admits a pointwise interpretation. Indeed, $x(t) = \mathbf{T}_t x(0)$ is a strong solution of that equation, if we consider it as an equation in X_{-1}, the space introduced in Definition 3.1. This is well known (see Nagel et al. [9, p. 19] or Salamon [14] or Weiss [18]).

EXAMPLE 1.1 (continued). Recall that our observation operator is given by

$$Cx = x(\xi),$$

where $\xi \in (0, \pi)$. C will now be regarded as an operator from $D(A)$ to X. Let $x \in L^2[0, \pi]$ be such that the left and right limits of x in ξ exist:

$$x(\xi^+) = \lim_{\tau \to 0, \tau > 0} x(\xi + \tau),$$

$$x(\xi^-) = \lim_{\tau \to 0, \tau > 0} x(\xi - \tau).$$

We don't assume these limits to be equal, i.e., x may have a jump in ξ. We want to show that $x \in D(C_L)$ and

$$C_L x = \tfrac{1}{2}[x(\xi^+) + x(\xi^-)]. \tag{4.5}$$

We only give the idea of the proof. Since $\mathbf{T}$ is analytic, the expression $C\mathbf{T}_t x$ makes sense for any $t > 0$. We want to show that $C\mathbf{T}_t x$ has a limit for $t \to 0$.

The semigroup $\mathbf{T}$ can be written as

$$(\mathbf{T}_t x)(\zeta) = \int_0^\pi G_t(\zeta, \theta) x(\theta) d\theta,$$

where the "Green function" G_t is given by

$$G_t(\zeta, \theta) = \frac{1}{(4\pi t)^{1/2}} \sum_{k \in \mathbf{N}} [e^{-(\zeta - \theta - 2k\pi)^2/4t} - e^{-(\zeta + \theta - 2k\pi)^2/4t}]$$

(see e.g. H. Dym and H. P. McKean [5, p. 67]). For t very small, all terms in the series giving G_t become negligible, except the term

$$G_t^0(\zeta, \theta) = \frac{1}{(4\pi t)^{1/2}} e^{-(\zeta - \theta)^2/4t},$$

which has a peak at $\zeta = \theta$. Because of the special form of C, we are only interested in the point $\zeta = \xi$. The integral $\int_0^\pi G_t^0(\xi, \theta) x(\theta) d\theta$ can be decomposed into an integral on a small neighborhood of ξ and an integral on the rest, the latter becoming negligible as $t \to 0$. The integral on the small neighborhood tends to the mean value of the left and right limit of x in ξ. By the definition of C_L we get that $x \in D(C_L)$ and (4.5) holds.

Example 1.2 (continued). For the semigroup $\mathbf{T}$ of this example, the domain of the generator A consists of the absolutely continuous functions $x \in X$ for which also $x' \in X$. The observation operators C_1, C_2 and C_3 are equivalent to

$C: D(A) \to \mathbf{C}$ defined by $Cx = x(0)$. A moment of thought will convince you that $x \in D(C_L)$ if and only if x has a Lebesgue point in 0, and then

$$C_L x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} x(\zeta) d\zeta .$$

Notice that $D(C_L)$ is not $\mathbf{T}$-invariant.

5. Invariance to perturbations

In the sequel we want to show that the Lebesgue extension of C is invariant to bounded perturbations of the semigroup.

LEMMA 5.1. *Let* $\mathbf{T}$ *be a semigroup on* X, *with generator* A, *let* $P \in \mathscr{L}(X)$ *and let* $\mathbf{S}$ *be the semigroup generated by* $A + P$.

Then for any $x \in X$

$$\lim_{\tau \to 0} \frac{1}{\tau} (\mathbf{S}_\tau x - \mathbf{T}_\tau x) = Px. \tag{5.1}$$

PROOF. We have for $\tau > 0$

$$\frac{1}{\tau} (\mathbf{S}_\tau x - \mathbf{T}_\tau x) = \frac{1}{\tau} \int_0^{\tau} \mathbf{T}_{\tau-\sigma} P \mathbf{S}_\sigma x \, d\sigma. \tag{5.2}$$

We decompose

$$\mathbf{T}_{\tau-\sigma} P \mathbf{S}_\sigma x - Px = \mathbf{T}_{\tau-\sigma} P(\mathbf{S}_\sigma x - x) + (\mathbf{T}_{\tau-\sigma} - I)Px.$$

We denote

$$M = \max_{t \in [0,1]} \| \mathbf{T}_t P \| .$$

Let $x \in X$ be fixed. Let $\varepsilon > 0$. Let $\delta_\varepsilon \in (0, 1]$ be such that for $t \in [0, \delta_\varepsilon]$,

$$\| \mathbf{S}_t x - x \| \leqq \frac{\varepsilon}{2M} \quad \text{and} \quad \| \mathbf{T}_t Px - Px \| \leqq \frac{\varepsilon}{2} .$$

Then for $\tau \in (0, \delta_\varepsilon]$ and $\sigma \in [0, \tau]$

$$\| \mathbf{T}_{\tau-\sigma} P \mathbf{S}_\sigma x - Px \| \leqq \varepsilon,$$

whence

$$\left\| \frac{1}{\tau} \int_0^{\tau} \mathbf{T}_{\tau-\sigma} P \mathbf{S}_\sigma x \, d\sigma - Px \right\| \leqq \varepsilon.$$

By (5.2) we get that (5.1) holds. □

THEOREM 5.2. *Let X and Y be Banach spaces, let $\mathbf{T}$ be a semigroup on X, with generator A, let $P\in\mathscr{L}(X)$, let $\mathbf{S}$ be the semigroup generated by $A+P$ and let $C\in\mathscr{L}(X_1, Y)$.*

Then the Lebesgue extension of C with respect to $\mathbf{S}$ is the same as with respect to $\mathbf{T}$.

PROOF. Let C_L be the Lebesgue extension of C with respect to $\mathbf{T}$ and let C_L' be the Lebesgue extension with respect to $\mathbf{S}$. Let $x\in X$. Let us denote $z(t)=\mathbf{S}_t x-\mathbf{T}_t x$. Then

$$C_L'x=\lim_{\tau\to 0}\left\{C\frac{1}{\tau}\int_0^\tau \mathbf{T}_\sigma x\,d\sigma+C\frac{1}{\tau}\int_0^\tau z(\sigma)d\sigma\right\}, \tag{5.3}$$

if the above limit exists (i.e., if $x\in D(C_L')$).

Subtracting the equalities

$$(A+P)\int_0^\tau \mathbf{S}_\sigma x\,d\sigma=\mathbf{S}_\tau x-x,$$

$$A\int_0^\tau \mathbf{T}_\sigma x\,d\sigma=\mathbf{T}_\tau x-x,$$

and dividing by $\tau>0$, we get that

$$A\frac{1}{\tau}\int_0^\tau z(\sigma)d\sigma=\frac{1}{\tau}(\mathbf{S}_\tau x-\mathbf{T}_\tau x)-P\frac{1}{\tau}\int_0^\tau \mathbf{S}_\sigma x\,d\sigma.$$

The above formula, together with Lemma 5.1, yields

$$\lim_{\tau\to 0}A\frac{1}{\tau}\int_0^\tau z(\sigma)d\sigma=Px-Px=0.$$

On the other hand, it is clear that

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau z(\sigma)d\sigma=0.$$

The last two equalities, together with the definition of the norm on X_1, imply that

$$\lim_{\tau\to 0}\left\|\frac{1}{\tau}\int_0^\tau z(\sigma)d\sigma\right\|_1=0.$$

Because of the continuity of C on X_1, the second term on the right-hand side of (5.3) tends to 0. Therefore, the limit in (5.3) exists if and only if the limit in (4.1) giving C_L exists, and then the two limits are equal. □

REMARK 5.3. In the conditions of Definition 4.1, let $s \in \rho(A)$ and $x \in D(C_L)$. Then

$$C_L x = \lim_{\tau \to 0} C \frac{e^{-s\tau}\mathbf{T}_\tau - I}{\tau} (A - sI)^{-1} x. \tag{5.4}$$

If x is not in $D(C_L)$ then the limit in (5.4) is not convergent. This follows immediately from (4.2) and Theorem 5.2, with $P = -sI$. In fact, (5.4) can be obtained also by a direct computation, without using Theorem 5.2.

REMARK 5.4. In the conditions of Theorem 5.2, if C is an admissible observation operator for $\mathbf{T}$ and some p, then it is admissible also for $\mathbf{S}$ and p. This will be proved in another paper.

6. Spaces of admissible C and duality

First we slightly reformulate the definition of admissibility given in the introduction. The change is that, in view of Remark 3.7, we restrict our attention to operators in $\mathscr{L}(X_1, Y)$. That restriction enables us to consider spaces of admissible observation operators.

DEFINITION 6.1. Let X and Y be Banach spaces, let $\mathbf{T}$ be a semigroup on X with generator A, let $C \in \mathscr{L}(X_1, Y)$ and let $p \in [1, \infty]$. Then C is an *admissible* observation operator for X, Y, $\mathbf{T}$ and p, if for some (and hence any) $T > 0$ there is some $K \geqq 0$ such that

$$\| C\mathbf{T}_t x \|_{L^p([0,T],Y)} \leqq K \| x \|, \qquad \forall x \in D(A). \tag{6.1}$$

The space $\mathscr{C}_p(X, Y, \mathbf{T})$ is the vector space of all admissible observation operators C for X, Y, $\mathbf{T}$ and p, with the norm $|\!|\!| C |\!|\!|_p$ given by the smallest possible $K \geqq 0$ for which (6.1) holds.

When Y is just $\mathscr{K}$, the field of the scalars ($\mathbf{R}$ or $\mathbf{C}$), we denote $c_p(X, \mathbf{T}) = \mathscr{C}_p(X, \mathscr{K}, \mathbf{T})$. We usually write $\mathscr{C}_p$ and c_p, without the arguments, when there is no danger of confusion.

REMARK 6.2. It is not difficult to verify, using the representation theorem 3.3, that $\mathscr{C}_p$ is complete. Indeed, a Cauchy sequence in $\mathscr{C}_p$ determines a convergent sequence of families of output maps, and the limit family admits a representation by an observation operator.

REMARK 6.3. Let C_L be the Lebesgue extension of C. Then C is admissible for $\mathbf{T}$ and p if and only if for any $x \in X$ and a.e. $t \geqq 0$, $\mathbf{T}_t x \in D(C_L)$ and $C_L \mathbf{T}_t x$, as a function of t, is in L^p_{loc}. This follows from Theorem 4.5 in one direction and from the closed graph theorem in the other.

REMARK 6.4. The following inclusions are immediate

$$\mathscr{L}(X, Y) \subset \mathscr{C}_p \subset \mathscr{L}(X_1, Y),$$

$$X^* \subset c_p \subset X_1^*,$$

$$\mathscr{C}_{p_1} \subset \mathscr{C}_{p_2} \qquad \text{for } p_1 \geqq p_2,$$

all with continuous embedding. It is further clear that a necessary condition for $C \in \mathscr{C}_p$ is

$$vC \subset c_p, \qquad \forall v \in Y^* \tag{6.2}$$

(this condition is not sufficient; see Remark 6.11 below).

There is one case when the determination of $\mathscr{C}_p$ is very easy, namely when $p = \infty$.

PROPOSITION 6.5. *Let X, Y and $\mathbf{T}$ be as in Definition 6.1. Then*

$$\mathscr{C}_\infty = \mathscr{L}(X, Y).$$

PROOF. Let $C \in \mathscr{C}_\infty$, i.e., (6.1) holds with $p = \infty$. For $x \in D(A)$, $C\mathbf{T}_t x$ is continuous in t, so

$$\| Cx \| \leqq \max_{t \in [0,T]} \| C\mathbf{T}_t x \| .$$

Together with (6.1) this gives that $C \in \mathscr{L}(X, Y)$. □

There is much analogy between the theory of admissible control operators, as presented in Weiss [18], and the theory of admissible observation operators, as presented here. This article has been written in such a way as to make this analogy conspicuous. However, there are also important differences. The notion of Lebesgue extension has, to my knowledge, no counterpart for control operators. On the other hand, the results concerning control operators for invertible semigroups, which appear in Weiss [18, §4], have no counterpart for observation operators. The representation theorem for abstract linear control systems (see Weiss [18, §3]) is proved only for $p < \infty$, while its counterpart, Theorem 3.3 here, is proved for any p. The counterpart of Proposition 6.5 here (see Weiss [18, §4]) holds only for reflexive X.

The duality between admissible control operators and admissible observation operators has been discussed in Curtain and Pritchard [2], Dolecki and Russell [4], Pritchard and Wirth [13], Salamon [16] and others. In the sequel we want to contribute to that subject by giving a precise formulation of the duality relationships between the spaces of admissible observation operators and the spaces of admissible control operators associated to a semigroup. First we recall a definition from Weiss [18, §4].

DEFINITION 6.6. Let U and X be Banach spaces, let $\mathbf{T}$ be a semigroup on X, let $B \in \mathscr{L}(U, X_{-1})$ and let $p \in [1, \infty]$. Then B is an *admissible* control operator for U, X, $\mathbf{T}$ and p, if for some (and hence any) $T > 0$ and for any $v \in L^p([0, T], U)$

$$\int_0^T \mathbf{T}_t Bv(t)dt \in X. \tag{6.3}$$

The space $\mathscr{B}_p(U, X, \mathbf{T})$ is the vector space of all admissible control operators B for $U, X, \mathbf{T}$ and p, with the norm $|||B|||_p$ given by the norm of the operator on the left-hand side of (6.3) (which is bounded by the closed graph theorem).

In the sequel we consider that a number $T > 0$ is fixed and is used in both Definitions 6.1 and 6.6 to define the norms on $\mathscr{C}_p$ and on $\mathscr{B}_p$.

For more information about admissible control operators we refer to Weiss [18] and the references given there. Here we shall need the following remark.

REMARK 6.7. For $p < \infty$ we have the following characterization of admissible control operators: $B \in \mathscr{L}(U, X_{-1})$ is admissible if and only if for any step function $v : [0, T] \to U$

$$\left\| \int_0^T \mathbf{T}_t Bv(t)dt \right\| \leqq K \cdot \| v \|_{L^p([0,T],U)}.$$

The norm $|||B|||_p$ is given by the smallest possible $K \geqq 0$ for which the above inequality holds.

Indeed, if v is a step function then for any $B \in \mathscr{L}(U, X_{-1})$ the integral on the left-hand side of (6.3) is in X, and if the above inequality holds then, by the density of step functions in L^p for $p < \infty$, this integral defines a bounded linear operator from L^p to X.

With the notation of Definition 6.1, $\mathbf{T}^*$ is a (not necessarily strongly continuous) semigroup on any of the spaces X^*, X_1^* and X_{-1}^*. If $\mathbf{T}^*$ is strongly continuous, then it is easy to check that

$$X_1^* = (X^*)_{-1}, \qquad X_{-1}^* = (X^*)_1.$$

The semigroup $\mathbf{T}^*$ is strongly continuous if and only if X_{-1}^* is dense in X^*, and this happens if and only if X^* is dense in X_1^*. This follows from a theorem in Pazy [10, p. 39]. A sufficient condition for $\mathbf{T}^*$ to be strongly continuous is that X is reflexive (see e.g. Pazy [10, p. 41]).

In the proof of the duality theorem (Theorem 6.9 below) we shall need the following lemma.

LEMMA 6.8. *Let Z be a Banach space, let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$ and let $T > 0$. Let S be the unit ball of $L^p([0, T], Z)$. Then for any $y \in L^q([0, T], Z^*)$*

$$\| y \| = \sup_{v \in S} \left| \int_0^T \langle v(t), y(t) \rangle dt \right| . \tag{6.4}$$

Let S' be the unit ball of $L^q([0, T], Z^*)$. Then for any $v \in L^p([0, T], Z)$

$$\| v \| = \sup_{v \in S'} \left| \int_0^T \langle v(t), y(t) \rangle dt \right| . \tag{6.5}$$

It has to be pointed out that generally $L^q([0, T], Z^*)$ is not the dual of $L^p([0, T], Z)$, unless $p < \infty$ and Z has the Radon–Nikodym property; see Diestel and Uhl [3, p. 98]. The proof of (6.4) can be found in [3, p. 97], where the unnecessary assumption $p < \infty$ is made. The equality (6.5) follows from (6.4) and the fact that y can be regarded as an element of $L^p([0, T], Z^{**})$, with the same norm.

THEOREM 6.9. *Let U, X, Y be Banach spaces, let $\mathbf{T}$ be a strongly continuous semigroup on X such that $\mathbf{T}^*$ is also strongly continuous and let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then we have the following duality relations:*

(i) *For any $C \in \mathscr{L}(X_1, Y)$,*

$$C \in \mathscr{C}_p(X, Y, \mathbf{T}) \Leftrightarrow C^* \in \mathscr{B}_q(Y^*, X^*, \mathbf{T}^*),$$

$$|||C|||_p = |||C^*|||_q.$$

(ii) *For any $B \in \mathscr{L}(U, X_{-1})$, if $p < \infty$ or if X is reflexive,*

$$B \in \mathscr{B}_p(U, X, \mathbf{T}) \Leftrightarrow B^* \in \mathscr{C}_q(X^*, U^*, \mathbf{T}^*),$$

$$|||B|||_p = |||B^*|||_q.$$

(iii) *For any* $B \in \mathcal{L}(U, X_{-1})$,

$$B \in \mathcal{B}_\infty(U, X, \mathbf{T}) \Rightarrow B^* \in \mathcal{C}_1(X^*, U^*, \mathbf{T}^*),$$

$$|||B|||_\infty = |||B^*|||_1.$$

The equalities of norms stated above should be understood in the following sense: If the operators belong to the corresponding spaces, then their norms are equal.

PROOF. (i) Suppose $C \in \mathcal{C}_p$. Let $x \in D(A)$, let $y \in L^q([0, T], Y^*)$ and denote

$$z_y = \int_0^T \mathbf{T}_t^* C^* y(t) dt,$$

so $z_y \in X_1^*$. Then, using the Hölder inequality and Definition 6.1, we have

$$\begin{aligned} |\langle x, z_y \rangle| &= \left| \int_0^T \langle C\mathbf{T}_t x, y(t) \rangle dt \right| \\ &\leqq \| C\mathbf{T}_t x \|_{L^p([0,T],Y)} \cdot \| y \| \\ &\leqq |||C|||_p \cdot \| x \| \cdot \| y \| . \end{aligned}$$

This shows that the functional z_y can be extended continuously to all of X, i.e., $z_y \in X^*$ and $\| z_y \| \leqq |||C|||_p \cdot \| y \|$. This implies that $C^* \in \mathcal{B}_q$ and $|||C^*|||_q \leqq |||C|||_p$.

Conversely, suppose $C^* \in \mathcal{B}_q$. Let $x \in D(A)$. Let S' denote the unit ball of $L^q([0, T], Y^*)$. Then using (6.5) we have

$$\begin{aligned} \| C\mathbf{T}_t x \|_{L^p([0,T],Y)} &= \sup_{y \in S'} \left| \int_0^T \langle C\mathbf{T}_t x, y(t) \rangle dt \right| \\ &= \sup_{y \in S'} \left| \left\langle x, \int_0^T \mathbf{T}_t^* C^* y(t) dt \right\rangle \right| \\ &\leqq |||C^*|||_q \cdot \| x \| . \end{aligned}$$

By looking at Definition 6.1 we conclude that $C \in \mathcal{C}_p$ and $|||C|||_p \leqq |||C^*|||_q$. This finishes the proof of (i).

(ii) Suppose $B \in \mathcal{B}_p$. Let $z \in D(A^*)$. Let S denote the unit ball of $L^p([0, T], U)$. Then using (6.4) we get, after a computation similar to that in the second part of the proof of (i), that

$$\| B^* \mathbf{T}_t^* z \|_{L^q([0,T],U^*)} \leqq ||| B |||_p \cdot \| z \| .$$

By looking at Definition 6.1 we conclude that $B^* \in \mathscr{C}_q$ and $||| B^* |||_q \leqq ||| B |||_p$.

Conversely, suppose $B^* \in \mathscr{C}_q$. Let $z \in D(A^*)$, let $v \in L^p([0, T], U)$ and denote $x_v = \int_0^T \mathbf{T}_t B v(t) dt$, so $x_v \in X_{-1}$. Then, using the Hölder inequality and Definition 6.1, we can get (see the first part of the proof of (i) for the idea) that

$$(6.6) \qquad | \langle x_v, z \rangle | \leqq ||| B^* |||_q \cdot \| z \| \cdot \| v \| .$$

If X is reflexive, then we continue as follows. Inequality (6.6) shows that x_v can be extended to a continuous functional on X^*, whence $x_v \in X$ and $\| x_v \| \leqq ||| B^* |||_q \cdot \| v \|$. That implies that $B \in \mathscr{B}_p$ and $||| B |||_p \leqq ||| B^* |||_q$.

If X is not assumed to be reflexive but $p < \infty$, then from (6.6) we proceed as follows. For any step function v we have that $x_v \in X$ and (6.6) implies $\| x_v \| \leqq ||| B^* |||_q \cdot \| v \|$. By Remark 6.7, B is admissible and $||| B |||_p \leqq ||| B^* |||_q$.

(iii) In the first part of the proof of (ii) we did not use that $p < \infty$ or that X is reflexive, so we have already proved there the implication $B \in \mathscr{B}_\infty \Rightarrow B^* \in \mathscr{C}_1$, as well as the inequality $||| B |||_\infty \geqq ||| B^* |||_1$. It follows that if $B \in \mathscr{B}_\infty$ then (6.6) holds, implying $||| B |||_\infty \leqq ||| B^* |||_1$, which finishes the proof of (iii). □

Remark 6.10. I don't know if the converse of the implication in (iii) of Theorem 6.9 holds (unless X is reflexive, so (ii) applies). It is not difficult to prove the following substitute. For any $B \in \mathscr{L}(U, X_{-1})$, we have that $B^* \in \mathscr{C}_1(X^*, U^*, \mathbf{T}^*)$ if and only if (6.3) holds for any regulated function v. A function $v: [0, T] \to U$ is called *regulated* if it is a uniform limit of step functions, or equivalently, if it has left and right limits in each point of its domain (with the obvious modification for endpoints). The proof is similar to the proof of (ii) above, but we need a strengthening of (6.4) for $p = \infty$: This equality remains true if we replace S by its intersection with the space of regulated functions.

Remark 6.11. Using the duality theorem, we can translate various negative results appearing in Weiss [18, §5], to get negative results about observation operators. We mention the following results obtained in this way:

(a) X^* may be a nondense subspace of c_p. Hence, in general $\mathscr{C}_p$ can not be obtained as the completion of $\mathscr{L}(X, Y)$ with respect to $||| \cdot |||_p$.
(b) The condition (6.2) is not sufficient for C to be admissible. It follows that for X, Y and $\mathbf{T}$ fixed, there is generally no Banach space $Z \subset X$ with the property that $\mathscr{C}_p(X, Y, \mathbf{T}) = \mathscr{L}(Z, Y)$. (Indeed, this would imply

$Z^* = \mathcal{c}_p$ and, by the uniform boundedness principle, that (6.2) is sufficient for admissibility.)

7. Diagonal semigroups on l^2

In this section we consider X, the state space, to be the Hilbert space l^2, and the semigroup $\mathbf{T}$ to be *diagonal*, i.e.,

$$(\mathbf{T}_t x)_k = e^{\lambda_k t} x_k, \qquad \forall k \in \mathbf{N}, \tag{7.1}$$

where x_k denotes the k-th component of x. The numbers λ_k, the eigenvalues of the generator A, satisfy

$$\sup_{k \in \mathbf{N}} \operatorname{Re}(\lambda_k) = -\sigma < 0. \tag{7.2}$$

The above restriction is unessential; it is made only for convenience (if it is not satisfied, we can shift A until (7.2) gets satisfied). The output space Y will be $\mathbf{C}$ (scalar output). We identify elements $c \in X_1^*$ with sequences (c_k) such that $\langle x, c \rangle = \sum_{k=1}^{\infty} x_k \overline{c_k}$ for any $x \in X_1$.

First we characterize the admissible observation operators for the above X, Y, $\mathbf{T}$ and $p = 2$. This characterization, a Carleson measure criterion, follows by duality from the corresponding criterion for control operators, which is due to Ho and Russell [7] and Weiss [19].

For $h > 0$ and $\omega \in \mathbf{R}$ we denote

$$R(h, \omega) = \{ z \in \mathbf{C} \mid 0 \leqq \operatorname{Re}(z) \leqq h,\ |\operatorname{Im}(z) - \omega| \leqq h \}.$$

$R(h, \omega)$ is a rectangle in the right complex half-plane, touching the imaginary axis.

We say that a complex sequence (c_k) satisfies the *Carleson measure criterion* for the sequence (λ_k), if for any $h > 0$ and any $\omega \in \mathbf{R}$

$$\sum_{-\lambda_k \in R(h,\omega)} |c_k|^2 \leqq M \cdot h, \tag{7.3}$$

where $M > 0$ is independent of h and ω.

Proposition 7.1. *For* $\mathbf{T}$ *given by* (7.1) *and satisfying* (7.2), $\mathcal{c}_2$ *(as introduced in Definition* 6.1) *is the space of sequences satisfying the Carleson measure criterion* (7.3).

Proof. By the duality theorem 6.9, $c \in X_1^*$ belongs to $\mathcal{c}_2$ if and only if c^* belongs to $\mathcal{B}_2$ (with respect to $\mathbf{T}^*$). $\mathbf{T}^*$ is given by (7.1), but with $\overline{\lambda_k}$ instead of λ_k.

According to the result in Weiss [19], $c^* \in \mathscr{B}_2$ if and only if $\overline{c^*}$ is a sequence satisfying the Carleson measure criterion for the sequence $(\overline{\lambda_k})$ (which is the same as for (λ_k)). Hence $c \in \mathscr{C}_2$ if and only if it is a sequence satisfying (7.3). □

The following proposition is helpful for evaluating $c_L x$ when $\mathbf{T}$ is a self-adjoint diagonal semigroup on $X = l^2$ and $c \in X_1^*$.

PROPOSITION 7.2. *Let* $\mathbf{T}$ *be given by* (7.1) *on* $X = l^2$ *and assume*

$$0 > \lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots. \tag{7.4}$$

Let $x \in X$ *and* $c \in X_1^*$. *If*

$$\sum_{k=1}^{\infty} c_k x_k = S, \tag{7.5}$$

then $x \in D(c_L)$ *and* $c_L x = S$.

Note that we didn't assume the series in (7.5) to be absolutely convergent, so the sum S could depend on the order of the terms. This ambiguity is eliminated by the ordering requirement (7.4).

PROOF. If the sequence (λ_k) is bounded then $A \in \mathscr{L}(X)$, so $X_1 = D(c_L) = X$ and the proof is finished. Assume $|\lambda_k| \to \infty$. According to (4.2) we have to check that

$$\lim_{\tau \to 0} c \, \frac{\mathbf{T}_\tau - I}{\tau} A^{-1} x = S. \tag{7.6}$$

We have

$$c \, \frac{\mathbf{T}_\tau - I}{\tau} A^{-1} x = \sum_{k=1}^{\infty} c_k \frac{e^{\lambda_k \tau} - 1}{\lambda_k \tau} x_k \tag{7.7}$$

(the series above is absolutely convergent, because $A^{-1}x \in X_1$ and $c \in X_1^*$). Let us denote for any $k \in \mathbf{N}$ and any $\tau > 0$

$$p_k(\tau) = \frac{e^{\lambda_k \tau} - 1}{\lambda_k \tau} - \frac{e^{\lambda_{k+1} \tau} - 1}{\lambda_{k+1} \tau}, \qquad s_k = \sum_{j=1}^{k} c_j x_j.$$

Then (7.4) implies $p_k(\tau) \geqq 0$ and $|\lambda_k| \to \infty$ implies

$$\lim_{\tau \to 0} \sum_{k=n}^{\infty} p_k(\tau) = 1, \qquad \forall n \in \mathbf{N}. \tag{7.8}$$

Summation by parts in (7.7) gives (using (7.5) and the fact that $|\lambda_k| \to \infty$)

$$c\,\frac{\mathbf{T}_\tau - I}{\tau}\,A^{-1}x = \sum_{k=1}^{\infty} p_k(\tau)s_k .$$

Let $\varepsilon > 0$. Choose $n \in \mathbf{N}$ such that for $k \geqq n$, $|s_k - S| \leqq \varepsilon$. We rewrite the last equality:

$$c\,\frac{\mathbf{T}_\tau - I}{\tau}\,A^{-1}x - S = \sum_{k=1}^{n-1} p_k(\tau)s_k + \sum_{k=n}^{\infty} p_k(\tau)(s_k - S) + S\left(\sum_{k=n}^{\infty} p_k(\tau) - 1\right).$$

Using (7.8) and the facts that $p_k(\tau) \geqq 0$ and $\lim_{\tau\to 0} p_k(\tau) = 0$, we can show that for τ sufficiently small, the right-hand side of the above equality is smaller, in absolute value, than 3ε. This proves (7.6). □

Remark 7.3. Consider the triple (A, B, C) introduced in Remark 4.9. Its transfer function $\mathbf{H}$ can be written in the form

$$\mathbf{H}(s) = C_L(s_0 I - A)^{-1}B + C(s_0 - s)(sI - A)^{-1}(s_0 I - A)^{-1}B,$$

where $s_0 \in \rho(A)$ is fixed. This shows that, avoiding computations involving C_L, we can compute $\mathbf{H}$ up to an additive constant. However, if we want to compute this additive constant then we have to evaluate the expression $C_L(s_0 I - A)^{-1}B$. The previous proposition is a tool for such evaluations, for a very special kind of A and C. See also the examples in [21].

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